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BORELAND JULIA DIRECTIONS OF MEROMORPHIC SCHRÖDER FUNCTIONS II

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Abstract. Let $R(w)$ be a non-linear rational function and s be a complex constant with $|s| > 1$. It is showed that for any solution $f(z)$ of the Schröder equation $f(sz) = R(f(z))$, Julia directions of $f(z)$ are also Borel directions of $f(z)$.

1. Introduction

Let $R(w)$ be a rational function of degree $p \geq 2$ and let s be a complex constant with $|s| > 1$. We consider the Schröder equation

$$f(sz) = R(f(z)). \tag{1.1}$$

We suppose that $R(0) = 0$ and $|R'(0)| > 1$. Let $f(z)$ be a meromorphic solution satisfying $f(0) = 0$ and $f'(0) = 1$. Such a solution exists uniquely, if $s = R'(0)$. The order of $f(z)$ equals to $\rho = \log p / \log |s| > 0$, and it holds $K_1 r^\rho < T(r, f) < K_2 r^\rho$ for some constants $0 < K_1 < K_2$, where $T(r, f)$ denotes the Nevanlinna characteristic function of $f(z)$, see e.g. [13, p.160]. Throughout this paper, we use standard notations in the Nevanlinna theory and the complex dynamics theory. The reader can see the definitions of the proximity function, the counting function and the characteristic function of a meromorphic function, and also the definitions of the Julia direction and the Borel direction, see, e.g. [6], [11]. For the complex dynamics theory, see e.g. [4], [5], [10]. Put $d_\omega = \{z = re^{i\omega} \ ; \ 0 \leq r < \infty\}$ and define

$$\Omega(\omega, \alpha) = \{z \ ; \ |\arg[z] - \omega| < \alpha\}, \quad \Omega_r(\omega, \alpha) = \Omega(\omega, \alpha) \cap \{|z| < r\}.$$

For a Schröder function $f(z)$, a ray d_ω is called *s-Julia direction* if $f(z)$ takes any value other than possible Picard exceptional value(s) of $f(z)$ in $\Omega(\omega, \alpha)$ for any α. It is also called *s-Borel direction of divergence type*, simply *sd*-Borel direction, for $f(z)$ if zeros $z_n(a; \Omega(\omega, \alpha))$ of $f(z) - a$ in $\Omega(\omega, \alpha)$, counted multiple zeros only once, satisfies

$$\sum_{n=1}^{\infty} \frac{1}{|z_n(a; \Omega(\omega, \alpha))|^\rho} = \infty \tag{1.2}$$

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for any a other than possible Picard exceptional value(s) of $f(z)$. We have shown that Julia directions and Borel directions for a solution $f(z)$ of (1.1) are s-Julia directions and sd-Borel directions, respectively [2]. Here we will prove the following theorem.

Theorem 1. *For a meromorphic solution $f(z)$ of* (1.1), *any s-Julia direction is also an sd-Borel direction.*

2. Proof of Theorem 1

Write $\mathbb{C} \cup \{\infty\}$ as $\hat{\mathbb{C}}$. For $R(w)$ in (1.1), there holds either one of the following four cases.

(i) There are $a, b \in \hat{\mathbb{C}}, a \neq b$, such that $R^{-1}(a) = \{a\}$ and $R^{-1}(b) = \{b\}$.
(ii) There are $a, b \in \hat{\mathbb{C}}, a \neq b$, such that $R^{-1}(a) = \{b\}$ and $R^{-1}(b) = \{a\}$.
(iii) There is the only value $a \in \hat{\mathbb{C}}$ such that $R^{-1}(a) = \{a\}$.
(iv) There are no such values. That is, for any $a \in \hat{\mathbb{C}}$ we have $R^{-1}(a)$ contains $a' \neq a$ and $R^{-1}(a')$ contains $a'' \neq a, a'$.

We define a set $E(R)$ as follows, $E(R) = \{a, b\}$ for the case (i) or (ii), $E(R) = \{a\}$ for (iii) and $E(R) = \emptyset$ for (iv) [10, p.32]. It is known that the set $E(R)$ coincides with the set of exceptional values of a rational function $R(z)$ which consists of those $a \in \hat{\mathbb{C}}$ such that the equation $R^n(z) = a$, $n \in \mathbb{N}$ have in totality a finite number of roots, where $R^n(z)$ denotes the n-th iteration of $R(z)$. We denote the Julia set of $R(w)$ by $\mathcal{J}_R$. The following lemma is well known [4, p. 145], [10, p. 30].

Lemma 1. *Let $\mathfrak{K} \subset \hat{\mathbb{C}} \setminus E(R)$ be a compact set. If D is a domain with $D \cap \mathcal{J}_R \neq \emptyset$. Then $R^n(D) \supset \mathfrak{K}$ for $n \geq n_0$ with some $n_0 \in \mathbb{N}$.*

We have proved the following lemma in [2].

Lemma 2. *Let d_ω be a Julia direction for the solution $f(z)$ of* (1.1). *Then $f(d_\omega) \cap \mathcal{J}_R \neq \emptyset$.*

Proof of Theorem 1. Write $s = |s|e^{2\pi\lambda i}$ with $\lambda \in [0, 1)$. If $\lambda \notin \mathbb{Q}$, then any direction is sd-Borel direction[2]. Hence Theorem 1 follows.

In the case $\lambda = \mu/\nu \in \mathbb{Q}$, considering $f(s^{2\nu}z) = R^{2\nu}(f(z))$ instead of (1.1), we are only concerned with (1.1) under the supposition $s > 1$.

When (i), (ii), or (iii) holds, $R(w)$ is conjugate to w^p, w^{-p}, or a polynomial, respectively (for the conjugacy, see [10, p. 24]). For the case (i), considering the conjugation and the conditions on (1.1), we see that the equation (1.1) reduces to the equation

$$f(pz) = -1 + (1 + f(z))^p = pf(z) + \cdots + f(z)^p. \tag{2.1}$$

The unique solution of (2.1) is given by $f(z) = e^z - 1$, which has s-Julia directions $d_{\pi/2}$ and $d_{3\pi/2}$. Obiously, these directions are also sd-Borel directions, which shows that Theorem 1 holds in this case. For the case (ii), we note that $R^2(w)$ is conjugate to w^{p^2}. We similarly consider the conjugation and the conditions on (1.1). Then we obtain the unique solution to the reduced equation for the case (ii), which gives the assertion. Therefore we have only to consider the cases (iii) and (iv).

Considering the conjugate form, we write $R(w) = a_1 w + \cdots + a_p w^p$ in case (iii) occurs. Let $L > 0$ be such that for $|w| \geq L$ we have

$$|a_p||w|^p > 2(|a_{p-1}||w|^{p-1} + \cdots + |a_1||w|) \quad \text{and} \quad |a_p||w|^p > 4|w|.$$

Then $|R(w)| > 2|w|$ and $|R^n(w)| > 2^n|w|$ for $|w| \geq L$.

Take $b \in \mathbb{C}$ for the case (iii), and $b \in \hat{\mathbb{C}}$ for (iv). Put $M_b = \max(|b|, L)$ for (iii). Obviously, we see that, if $c \in R^{-n}(b)$, then $|c| < M_b$. For, otherwise, we would have $|R^n(c)| \geq 2^n|c| \geq 2^n|b| > |b|$, a contradiction. We choose the compact set $\mathfrak{K}$ in Lemma 1 as $\mathfrak{K} = \{w \ ; \ |w| \leq M_b\}$ for (iii), and $\mathfrak{K} = \hat{\mathbb{C}}$ for (iv).

Let d_ω be a Julia direction of $f(z)$. Write a sector $\Omega(\omega, \alpha)$ simply as Ω. Put $\Omega^r = \Omega \cap \{r < |z| < s^2 r\}$. By Lemma 2, there is r_0 such that $f(\Omega^{r_0}) \cap \mathcal{J}_R \neq \emptyset$. Hence there is $n_0 \in \mathbb{N}$ satisfying $R^n(f(\Omega^{r_0})) \supset \mathfrak{K}$ for $n \geq n_0$, by Lemma 1. Therefore, if we write $r_1 = s^{n_0+1} r_0$, then

$$f(s^n \Omega^{r_1}) = R^n(f(\Omega^{r_1})) \supset \mathfrak{K} \supset R^{-n}(b). \tag{2.2}$$

Let $\mathcal{C}$ be the set of critical points of $R(w)$. It consists of $2p-2$ points, counted with multiplicity [10, p. 27]. Let n_{spatr} be the number of superattracting cycles for $R(w)$. Then $n_{\text{spatr}} \leq 2(p-1)$, [9] [10, p. 55]. Let $C_{\text{spatr}} = \{w_k \ ; \ w_k$ belongs to some superattracting cylce$\}$. Clearly, we have $C_{\text{spatr}} \subset \mathcal{O}^+(\mathcal{C}) = \bigcup_{n=1}^{\infty} R^n(\mathcal{C})$ (the orbit of $\mathcal{C}$). We divide the remaining part of the proof in three steps, namely $b \notin \mathcal{O}^+(\mathcal{C})$, $b \notin C_{\text{spatr}}$ and $b \in C_{\text{spatr}}$. Note that $0 \notin C_{\text{spatr}}$, since $0 = R(0)$, $|R'(0)| > 1$.

First we are concerned with the case $b \notin \mathcal{O}^+(\mathcal{C})$. Write $R^{-n}(b) = \{b_j^{(n)}; 1 \leq j \leq p^n\}$, with $b_j^{(n)} \neq b_{j'}^{(n)}$ if $j \neq j'$. By (2.2)

$$\overline{N}(s^n r_1, b, f; \Omega) = \sum_{j=1}^{p^n} \overline{N}\left(r_1, b_j^{(n)}, f; \Omega\right) \geq p^n \log s. \tag{2.3}$$

Let $s^n r_1 \leq r < s^{n+1} r_1$. Then from (2.3),

$$\begin{aligned} \int_{r_1}^{r} \frac{N(t, b, f; \Omega)}{t^{\rho+1}} dt &\geq \sum_{k=0}^{n-1} \int_{s^k r_1}^{s^{k+1} r_1} \frac{p^k \log s}{(s^{k+1} r_1)^\rho t} dt = \sum_{k=0}^{n-1} \frac{(\log s)^2}{p r_1^\rho} \\ &= \frac{(\log s)^2}{p r_1^\rho} \cdot n \to \infty, \quad r \to \infty \ (n \to \infty). \end{aligned} \tag{2.4}$$

Secondly we consider the case $b \notin C_{\text{spatr}}$. Suppose $c_1 \in \mathcal{C} \cap R^{-n_1}(b) \neq \emptyset$ for some n_1. Suppose $c_2 \in \mathcal{C} \cap R^{-n_2}(c_1) \neq \emptyset$ for some n_2. Further suppose this procedure could be repeated infinitely many. Since $\mathcal{C}$ is a finite set, there would be a super-attracting cycle, a contradiction. Therefore there is $c_k \in \mathcal{C} \cap R^{-n_k}(c_{k-1})$ such that $R^{-n}(c_k) \cap \mathcal{C} = \emptyset$ for any $n \geq 1$. Thus $d \in R^{-1}(c_k)$ does not belong to $\mathcal{O}^+(\mathcal{C})$, hence (2.4) holds for d, therefore also for $b = R^{m+1}(d), m = n_1 + \cdots + n_k$, with $s^{m+1}r$ for r in (2.4).

Finally we treat the case $b \in C_{\text{spatr}}$. Obviously C_{spatr} is a finite set. Hence, for sufficiently large m, there is some $d \in R^{-m}(b)$ with $d \notin C_{\text{spatr}}$. Thus (2.4) holds for d. Therefore (2.4) holds also for $b = R^m(d)$, with $s^m r$ for r in (2.4). □

3. An example

We consider the equation $f(sz) = (1-z)f(z)^2,\ s > 4$, which admits a solution

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{s^n}\right)^{2^{n-1}}.$$

We will show that for a large s the positive real axis is the only s-Julia direction for $f(z)$. However it is not an sd-Borel direction. This implies that Theorem 1 does not hold for solutions of equations $f(sz) = R(z, f(z))$.

We have that the order of $f(z)$ equals to $\rho = \log 2/\log s < 1/2$. Obviously $f(z) \to \infty$ as $z \to \infty$ with $\Re[z] \leq 0$.

Put $z = x + iy, y = ax(a = \tan\phi \neq 0)$ and $x = s^{n_0}t/\sqrt{1+a^2}$, $1/\sqrt{s} \leq t < \sqrt{s}$,

$$\left|1 - \frac{z}{s^n}\right|^2 = 1 - \frac{2x}{s^n} + \frac{(1+a^2)x^2}{s^{2n}} = 1 - \frac{2}{\sqrt{1+a^2}}\frac{s^{n_0}t}{s^n} + \left(\frac{s^{n_0}t}{s^n}\right)^2.$$

For $1 \leq n \leq n_0 - 1$, write $n = n_0 - k, 1 \leq k \leq n_0 - 1$. Then

$$1 - \frac{2}{\sqrt{1+a^2}}\frac{s^{n_0}t}{s^n} + \left(\frac{s^{n_0}t}{s^n}\right)^2 \geq -2\frac{s^{n_0}t}{s^n} + \left(\frac{s^{n_0}t}{s^n}\right)^2 = t^2 s^{2k}\left(1 - \frac{2}{t}\frac{1}{s^k}\right),$$
$$1 - \frac{2}{\sqrt{1+a^2}}\frac{s^{n_0}t}{s^n} + \left(\frac{s^{n_0}t}{s^n}\right)^2 \leq 1 + t^2 s^{2k}.$$

hence

$$L_{1,a}^{(n_0)}(t) = \sum_{k=1}^{n_0-1} \frac{1}{2^k}\log\left|1 - \frac{z}{s^{n_0-k}}\right|^2 \geq \sum_{k=1}^{n_0-1}\left\{\frac{2k}{2^k}\log s - \frac{4}{t}\frac{1}{(2s)^k} + \frac{\log t^2}{2^k}\right\}$$
$$= \left(3 - \frac{n_0+1}{2^{n_0-2}}\right)\log s - \frac{4}{t}\frac{1}{2s-1}, \qquad L_{1,a}^{(n_0)}(s^{1/4}) \geq 2\log s - 4s^{-5/4},$$
$$L_{1,a}^{(n_0)}\left(\frac{1}{\sqrt{1+a^2}}\right) \leq \sum_{k=1}^{n_0-1}\frac{1}{2^k}\log(1+s^{2k}) \leq \sum_{k=1}^{n_0-1}\frac{2k}{2^k}\log s + \log 2 \leq 4\log s + \log 2.$$

For $n = n_0$,

$$L_{2,a}^{(n_0)}(t) = \log\left(1 - \frac{2t}{\sqrt{1+a^2}} + t^2\right),$$

$$L_{2,a}^{(n_0)}(s^{1/4}) > 0, \qquad L_{2,a}^{(n_0)}\left(\frac{1}{\sqrt{1+a^2}}\right) = \log\frac{a^2}{1+a^2}.$$

For $n \geq n_0 + 1$, write $n = n_0 + k, k \geq 1$. Taking s so large that $2/\sqrt{s} < 1/2$, we get

$$\begin{aligned}
L_{3,a}^{(n_0)}(t) &= \sum_{k=1}^{\infty} 2^k \log\left|1 - \frac{z}{s^{n_0+k}}\right|^2 \geq \sum_{k=1}^{\infty} 2^k \log\left(1 - \frac{2}{\sqrt{1+a^2}}\frac{t}{s^k}\right) \\
&\geq \sum_{k=1}^{\infty} 2^k \left\{-\frac{4}{\sqrt{1+a^2}}\frac{t}{s^k}\right\} = -\frac{4t}{\sqrt{1+a^2}}\frac{2}{s-2}, \\
L_{3,a}^{(n_0)}&(s^{1/4}) \geq -16 s^{-3/4}, \\
L_{3,a}^{(n_0)}&\left(\frac{1}{\sqrt{1+a^2}}\right) \leq \sum_{k=1}^{\infty} 2^k \log\left(1 + \frac{1}{s^{2k}}\right) \leq \sum_{k=1}^{\infty}\left(\frac{2}{s^2}\right)^k = \frac{2}{s^2-2}.
\end{aligned}$$

Suppose there would be $a = \tan\phi > 0$ such that $f(z)$ assume some $b \in \mathbb{C} \setminus \{0\}$ at most finitely many times in $\Omega(0,\phi)$. Then $f(z)$ would omit three values $0, b, \infty$ in $\Omega(\phi/2, \phi/4) = \{|\arg[z] - \phi/2| < \phi/4\}$, and hence $\{f(s^n z)\}$ would be a normal family in

$$Q = \left\{z\ ;\ s^{-1/2} < |z| < s^{1/2},\ \frac{\phi}{4} < \arg[z] < \frac{3\phi}{4}\right\}.$$

Thus there would be a subsequence $\{n_\ell\}$ such that $f(s^{n_\ell} z)$ tends to either ∞ or a function $g(z)$ uniformly on any compact subset of Q. But, if we write $z_0 = e^{i\phi/2}$ and $a_1 = \tan(\phi/2)$, then for a large s

$$\begin{aligned}
4\log|f(s^{n_\ell} s^{1/4} z_0)| &= 2^{n_\ell}\{L_{1,a_1}(s^{1/4}) + L_{2,a_1}(s^{1/4}) + L_{3,a_1}(s^{1/4})\} \\
&\geq 2^{n_\ell}\{2\log s - 4s^{-5/4} - 16 s^{-3/4}\} \to \infty.
\end{aligned}$$

Once we fix s large enough that the inequality above holds. It suffices to consider the case a, and also a_1 are sufficiently small. Hence we have

$$4\log\left|f\left(\frac{s^{n_\ell}}{\sqrt{1+a_1^2}} z_0\right)\right| \leq 2^{n_\ell}\left\{4\log s + \log\frac{a_1^2}{1+a_1^2} + \frac{2}{s^2-2} + \log 2\right\} \to -\infty,$$

as $n_\ell \to \infty$, supposed that $4\log s + \log(a_1^2/(1+a_1^2)) + 2/(s^2-2) + \log 2 < 0$, a contradiction. Therefore $f(z)$ admits any finite value infinitely often in $\Omega(0,\phi)$ for any ϕ, hence the positive real axis d_0 is the only s-Julia direction for $f(z)$.

On the other hand, zeros $z_n(0,\Omega(0,\phi))$ of $f(z)$, in $\Omega(0,\phi)$, are $z = s^n$ with multiplicity 2^{n-1}. Hence we get

$$\sum_{n=1}^{\infty} \frac{1}{|z_n(0,\Omega(0,\phi))|^\rho} = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

in which multiple zeros are counted only once, and 0 is an exceptional value, which means that d_0 is not sd-Borel direction (though it is a Borel direction), and Theorem 1 does not hold in this case.

At the end of this section, we pose a note. Consider sd-Borel direction in the sense of Valiron, in which we count $z_n(0,\Omega(0,\phi))$ with multiplicity. Since we have

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{|z_n(0,\Omega(0,\phi))|^\rho} = \infty,$$

zero is not exceptional value for $f(z)$. Therefore d_0 is sd-Borel direction if we consider it in the sense of Valiron.

4. Some remarks

Existence of Julia directions was proved by use of “cercles de remplissage” [3]. Sauer [8] called such a Julia direction as *Milloux direction*. But a Julia direction need not necessarily be a Milloux direction [15].

On the other hand, on any Borel direction, there are centers of cercles de remplissage [7, p.160 Theorem X]. Thus, for Schröder functions, any Julia direction is a Milloux direction.

Recently, Zheng and others [1], [14] have newly introduced some singular directions for meromorphic functions.

Julia directions are known from the geometry of the Julia set of $R(w)$ [2]. Hence our theorem will give some tools for determining Borel directions for $f(z)$.

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